IJICT International Journal of Information & Communication Technology

ITRC

Volume 1- Number 1- March 2009 (pp. 3-12)

On the Distribution of the Sum of Independent Random Variables and Its Application

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Abstract— **An approximate analytical method for the evaluation of the cumulative distribution function (CDF) of the sum of L independent random variables (RVs) is presented. The proposed method is based on the convergent infinite series approach, which makes it possible to describe the CDF in the form of an infinite series. The computation of the coefficients of this series needs complicated integrations over the RV's probability density function (PDF). In some cases, the required integrations have closed-form in terms of confluent hypergeometric function and in other cases, the required integrations can not be analytically solved and have not a closed-form solution. In this paper, an approximation method for computation of the coefficients of the CDF series is presented that only needs the mean and the variance of the RV, so it has low computational complexity; it eliminates the need for calculation of complex functions and can be used as a unified tool for determining CDF of a sum of statistically independent RVs. To present an application for the developed approximation method, it is used to find the distribution of the sum of generalized Gamma (GG) RVs. The derived approximate expressions are used in the performance analysis of equal-gain combining (EGC) receivers operating over GG fading channels. The accuracy of the developed approximation method is verified by performing comparisons between exact existing results in the literature and computer simulations results.**

Keywords- Cumulative Distribution function, Generalized Gamma fading, Equal-gain combining, Convergent Infinite Series.

I. Introduction

The problem of finding the distribution of the sum of statistically independent random variables (RVs) is a well-recognized but cumbersome statistical task [1]. Such a problem occurs in several wireless applications. For example, in the performance analysis of equal-gain combining (EGC) receiver over fading channels, where the received faded signals are equally weighted, cophased, and then summed to form the resultant output. Since an analytical solution for this problem is very difficult to derive, the use of approximate solutions and union bound were proposed in the literature. In [2], an infinite series was derived for determining the cumulative distribution function (CDF) of the sum of Rayleigh distributed RVs. In [3], and [4] the distribution of such a sum was presented using saddle point integration for uniformly weighted RVs, and for arbitrary weights, respectively. In [5], the sum of Nakagami RVs was considered and an approximate probability density function (PDF) expression was derived for such a sum. In [6], accurate and simple closed-form approximations to the CDF and PDF of the sum of independent and identically distributed (i.i.d.) Rayleigh RVs were presented, while in [7] a closed-form union upper bound for the CDF of the weighted sum of independent Rayleigh RVs was derived. Very recently and with the aid of the well-known arithmetic–geometric mean inequality, in [8] a closed-form union upper-bound in terms of the complex Meier's G functions was presented for the distribution of the sum of independent generalized Gamma (GG) distributed RVs.

In this paper, an approximate analytical method for the computation of the CDF of a sum of independent, but not necessarily identically distributed RVs is developed based on convergent infinite series approach [2]. This approach makes it possible to describe the CDF of a sum of independent RVs as an infinite series. One of the main challenges for applying infinite series approach is the computation of the required coefficients in the CDF series, which needs complicated integration over the RV's distribution function. In some cases, for example Nakagami and Rician distributions, the required integrations have closed-form in terms of confluent hypergeometric function [9] [10]. The need for the evaluation of special complex functions as the confluent hypergeometric presents serious time overflow problems under some circumstances [11, Appendix B]; moreover, in [10] the Nakagami m-parameter was constrained to take integer values which are not true for real mobile radio environments. In other cases such as Weibull distribution, the required integrations can not be analytically solved and have not a closed-form solution. In this paper, an approximation method for computation of the coefficients of the CDF series is presented that avoids the calculation of complex functions and can be efficiently applied to practical wireless applications. This method only needs the mean and the variance of the RV's distribution; hence it eliminates the need for complicated integration over RV's distribution function while computing the required coefficients. This fact provides interesting features to the developed method. It has low computational complexity and can be used as a unified method for determining CDF of a sum of statistically independent RVs, especially where the required integration has not a closed-form solution or tabulated.

In order to show the general applicability of the proposed method, it is used to find the distribution of the sum of GG RVs. GG distribution is a generic model and covers the well-known fading models including Nakagami, Weibull and Rayleigh as special cases, the lognormal as a limiting case, and it can approximate Suzuki distribution [12][13], so it is a good candidate to show the applicability of the developed approximation method. To show the accuracy of the proposed method, the special cases of GG distribution are considered. The results derived based on the proposed method match accurately with the exact existing results related to these special cases. Moreover, for arbitrarily GG distribution parameters, the required quantities are numerically evaluated and are compared with those derived based on the developed approximate expressions. Results show the good accuracy of the proposed method for different GG distribution parameters.

Despite the ability of GG distribution to characterize so many different fading channel models, only a few performance study of diversity and specifically EGC receivers over GG fading channels have been presented in the literature [14]-[15] and [8]. In [14] the average symbol error probability of EGC receiver with coherent multilevel modulation schemes is obtained by employing a characteristic function based approach in independent GG fading channels. However, the solutions were presented in integral form. In [15] a canonical-form expression of the average symbol error probability for EGC receiver with M-ary orthogonal FSK under the assumption of independent GG faded branches was presented. In [8] union upper bounds for the outage and the average bit error probability were derived and were evaluated in terms of Meijer's G-functions.

In this paper a different approach is used for performance analysis of L-branch EGC receivers operating over GG fading channel. For such receivers, the proposed approximation method is used to derive the CDF and the PDF of the output signal-to-noise ratio (SNR). One straightforward usage of these important statistical functions is to evaluate the outage probability, the average probability of error and to study the effects of the different GG fading channel parameters on the performance of EGC receiver. The accuracy of the developed approximation method is verified by performing comparisons between existing upper bounds in the literature and computer simulations results.

The reminder of the paper is organized as follows. In section II, the convergent infinite series approach for deriving the CDF of the sum of independent RVs is discussed briefly. In section III, the proposed approximation method is introduced. Distribution of the sum of GG distributed RVs based on the proposed method is provided in section IV. Section V contains the accuracy analysis of the proposed approximation method. As a practical application, in section VI the developed method is used in the performance analysis of L-branch EGC receiver operating over GG fading channels. Finally, the main points are summarized in section VII.

II. THE CDF OF THE SUM OF INDEPENDENT RANDOM VARIABLES

Let $X = X_1 + X_2 + ... + X_L$ be the sum of L independent RVs $X_i, (i = 1,2,..., L)$, then the CDF of X, $F_X(x)$ can be computed within a determined accuracy as [2]:

$$F_X(x) = \Pr(X \le x) = \frac{1}{2} - \frac{2}{\pi} \sum_{\substack{n=1 \\ n \text{ is odd}}}^{\infty} \frac{A_n \sin(\tau_n)}{n} \quad (1)$$

with

$$A_n = \prod_{i=1}^{L} \sqrt{\left(E^2\{\cos(n\omega X_i)\} + E^2\{\sin(n\omega X_i)\}\right)} \quad (2)$$

$$\tau_n = \sum_{i=1}^{L} \tan^{-1}\left(\frac{E[\sin(n\omega(X_i - \varepsilon))]}{E[\cos(n\omega(X_i - \varepsilon))]}\right) \quad (3)$$

where $\varepsilon = x / L$ and $\omega = 2\pi / T$. Here, T is the period of the square wave used in deriving the series [2], and E [.] denotes expectation. By taking the first derivative of CDF with respect to x, the corresponding PDF can be obtained. To evaluate (2), it is required to determine A_n and θ_n or equivalently to determine $E\{\cos(n\omega X_l)\}$ and $E\{\sin(n\omega X_l)\}$. One possible approach is to evaluate the integrals after multiplying the Cosine or Sine term with the PDF of X_l, as follows:

$$E[\cos(n\omega X_l)] = \int_0^\infty \cos(n\omega x_l) f_{X_l}(x_l) dx_l \quad (4)$$

$$E[\sin(n\omega X_l)] = \int_0^\infty \sin(n\omega x_l) f_{x_l}(x_l) dx_l \quad (5)$$

One of the main challenges for applying infinite series approach is the computation of the required coefficients in the CDF series (i.e., A_n and θ_n in (3) and (4)), which needs the calculation of the $E\{\cos(n\omega X_l)\}$ and $E\{\sin(n\omega X_l)\}$ in (4) and (5). GG distribution was introduced by Stacy, back in 1962, as a generalization of the two-parameter Gamma distribution [17]. The PDF of the GG distributed RV X_l is given by [17, eq. (1)]:

$$f_{X_l}(x_l) = \frac{2v_l}{(\overline{\Omega}_l/m_l)^{m_l}\Gamma(m_l)} x_l^{2v_l m_l - 1} \exp\left(-\frac{m_l x_l^{2v_l}}{\overline{\Omega}_l}\right) \quad (6)$$

where $m_l \geq 1/2$ is the fading parameter, $v_l > 0$ is the shape parameter, $\overline{\Omega}_l$ is the average signal-to-noise (SNR) scaling parameter and $\Gamma(.)$ is the Gamma function. The lower and upper tails of the distributions in (16) can be adjusted by controlling the parameters m_l and v_l, respectively [13]. Consequently, the distribution in (16) is very generic since it covers several commonly used fading distributions as special or limiting cases: Rayleigh (for $v_l = 1$, $m_l = 1$), Nakagami (for $v_l = 1$), Weibull (for $m_l = 1$), and lognormal (for $m_l \to \infty, v_l \to 0$). The GG distribution also has the interesting property that it can model both amplitude and intensity fluctuations. By using (7), the mean μ_{X_l} and the variance $\sigma^2_{X_l}$ of GG distributed RV X_l can be described as:

$$\mu_{X_l} = E\{X_l\} = \left(\frac{\Omega_l}{m_l}\right)^{\frac{1}{2v_l}} \frac{\Gamma\left(m_l + \frac{1}{2v_l}\right)}{\Gamma(m_l)} \quad (7)$$

$$\sigma^2_{X_l} = \left(\frac{\Omega_l}{m_l}\right)^{\frac{1}{v_l}} \frac{\Gamma\left(m_l + \frac{1}{v_l}\right)}{\Gamma(m_l)} - \left(\frac{\Omega_l}{m_l}\right)^{\frac{1}{v_l}} \left\{\frac{\Gamma\left(m_l + \frac{1}{2v_l}\right)}{\Gamma(m_l)}\right\}^2 \quad (8)$$

For computation of $E\{\cos(n\omega X_l)\}$ and $E\{\sin(n\omega X_l)\}$ in the case of GG distribution, one needs to calculate the integrals of (4) and (5) considering $f_{X_l}(x_l)$ given in (6) as follows:

$$E[\cos(n\omega X_l)] = \frac{2v_l}{(\overline{\Omega}_l/m_l)^{m_l}\Gamma(m_l)} \int_0^\infty x_l^{2v_l m_l - 1} \cos(n\omega x_l) \exp\left(-\frac{m_l x_l^{2v_l}}{\overline{\Omega}_l}\right) dx_l \quad (9)$$

$$E[\sin(n\omega X_l)] = \frac{2v_l}{(\overline{\Omega}_l/m_l)^{m_l}\Gamma(m_l)} \int_0^\infty x_l^{2v_l m_l - 1} \sin(n\omega x_l) \exp\left(-\frac{m_l x_l^{2v_l}}{\overline{\Omega}_l}\right) dx_l \quad (10)$$

At first, we consider the integral in (9). To solve this integral, we can start with presenting the Cosine function as contour integral [18],

$$\cos(n\omega X_l) = \frac{\sqrt{\pi}}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} \frac{\Gamma(-s)}{\Gamma(\frac{1}{2}+s)} \left(\frac{n\omega x_l}{2}\right)^{2s} ds \quad (11)$$

By applying (11) in (4) and interchanging the order of integrations, we will have (12) in the bottom of this page. If we let $v_l = l_l / k_l$, where l_l and k_l are positive integers, and change the variable s with $l_l s$ and use the multiplication theorem, (12) can be written in terms of Meijer's *G*-function as is shown in (13) in top of the next page, where Meijer's *G*-function is defined as follows [18]:

$$G^{m,n}_{p,q}\left(z \left| \begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_p \end{matrix} \right.\right) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\prod_{j=1}^{n}\Gamma(1 - a_j + s)}{\prod_{j=m+1}^{q}\Gamma(1 - b_j + s)\prod_{j=n+1}^{p}\Gamma(a_j - s)} z^s ds \quad (14)$$

By using the same approach, $E\{\sin(n\omega X_l)\}$ can be described as (15). Note that Meijers *G*-function is a standard built-in function in well-known mathematical software packages, such as MAPLE and MATHEMATICA.

It can be shown that for the special case of $v_l = 1$ ($l_l = k_l = 1$) (13) and (15) reduce to equations (12) and (13) of [10] for Nakagami distribution. To verify this fact, we first consider (13) and insert $v_l = 1$ ($l_l = k_l = 1$). So, (13) converts to:

$$E[\cos(n\omega X_l)] = \frac{\sqrt{\pi}}{\Gamma(m_l)} G^{1,1}_{1,2}\left[\left(\frac{n^2\omega^2\overline{\Omega}_l}{4m_l}\right)^2 \left| \begin{matrix} 1 - m_l \\ 0; \frac{1}{2} \end{matrix} \right.\right] \quad (16)$$

Considering the relation between Meijer's *G*-function and hypergeometric function

$$E[\cos(n\omega X_l)] = \frac{2 v_l}{(\overline{\Omega}_l / m_l)^{m_l}\,\Gamma(m_l)} \int_{\sigma-j\infty}^{\sigma+j\infty} \frac{\Gamma(-s)}{\Gamma(\frac{1}{2}+s)} (\frac{n\omega}{2})^{2s} \left\{ \int_0^{\infty} x_l^{2(v_l m_l + s)-1} \exp(-\frac{m_l x_l^{2v_l}}{\overline{\Omega}_l})\, dx_l \right\} ds \quad (12)$$

$$= \frac{\sqrt{\pi}}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} ((\frac{\overline{\Omega}_l}{m_l})^{\frac{1}{v_l}} \frac{(n\omega)^2}{4})^s \frac{\Gamma(-s)\Gamma(m_l + s/v_l)}{\Gamma(\frac{1}{2}+s)} ds$$

$$E[\cos(n\omega X_l)] = \frac{\sqrt{\pi l_l}\, k_l^{(m_l-1/2)}}{\Gamma(m_l)(2\pi)^{\frac{k_l-1}{2}}} G_{k_l, 2l_l}^{l_l, k_l}\left[\left(\frac{n\omega}{2l_l}\right)^{2l_l} \left(\frac{k_l \overline{\Omega}_l}{m_l}\right)^{k_l} \middle| \begin{matrix} \left\{1-\frac{m_l}{k_l}-\frac{j-1}{k_l}\right\}_{j=1,\ldots,k_l} \\ \left\{\frac{j-1}{l_l}\right\}_{j=1,\ldots,l_l} ; \left\{2-\frac{j}{l_l}+\frac{1}{2l_l}\right\}_{j=l_l+1,\ldots,2l_l} \end{matrix} \right] \quad (13)$$

$$E[\sin(n\omega X_l)] = \frac{\sqrt{\pi l_l}\, k_l^{(m_l-1/2)}}{\Gamma(m_l)(2\pi)^{\frac{k_l-1}{2}}} G_{k_l, 2l_l}^{l_l, k_l}\left[\left(\frac{n\omega}{2l_l}\right)^{2l_l} \left(\frac{k_l \overline{\Omega}_l}{m_l}\right)^{k_l} \middle| \begin{matrix} \left\{1-\frac{m_l}{k_l}-\frac{j-1}{k_l}\right\}_{j=1,\ldots,k_l} \\ \left\{\frac{j-1/2}{l_l}\right\}_{j=1,\ldots,l_l} ; \left\{2-\frac{j+1}{l_l}\right\}_{j=l_l+1\ldots,2l_l} \end{matrix} \right] \quad (15)$$

$${}_pF_q(a;b;z) = \frac{\Gamma(b)}{\Gamma(a)} G_{p,1+q}^{1,p}\left[-z \middle| \begin{matrix} 1-a \\ 0;1-b \end{matrix} \right] \quad (17)$$

where ${}_1F(a;b;z)$ is the confluent hypergeometric function defined as [18, eq. (9.210.1)].

$${}_1F_1(a;b;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n n} z^n \quad (18)$$

Where $(a)_0 = 1$ and $(a)_n = \Gamma(a+n)/\Gamma(n)$. Equation (16) can be described in terms of hypergeometric function as follow

$$E[\cos(n\omega X_l)] = {}_1F_1\left(m_l; \frac{1}{2}; -\frac{n^2\omega^2\overline{\Omega}_l}{4m_l}\right) \quad (19)$$

which is similar to the result which was presented in [10, eq. (12)]. In the same manner, with $v_l = 1$ ($l_l = k_l = 1$), (15) changed to:

$$E[\sin(n\omega X_l)] = \frac{\sqrt{\pi}}{\Gamma(m_l)} G_{1,2}^{1,1}\left[\frac{n^2\omega^2\overline{\Omega}_l}{4m_l} \middle| \begin{matrix} 1-m_l \\ \frac{1}{2};0 \end{matrix} \right] \quad (20)$$

Considering (18) and the properties of Meijer's G - function [18] which is presented in (21), equation (19) can be described as (22) in top of the next page which is similar to the result that was presented in [10, eq. (13)].

$$G_{p,q}^{m,n}\left[z \middle| \begin{matrix} a+\alpha \\ b+\alpha \end{matrix} \right] = z^{\alpha} G_{p,q}^{m,n}\left[z \middle| \begin{matrix} a \\ b \end{matrix} \right] \quad (21)$$

Although, (13) and (15) give exact closed-form results for calculating $E\{\cos(n\omega X_l)\}$ and $E\{\sin(n\omega X_l)\}$, since the derived expressions are presented in terms of complex Meijer's G- function, results have limited usage. In the next section, we develop a new approximation method for calculation of the expressions for $E\{\cos(n\omega X_l)\}$ and $E\{\sin(n\omega X_l)\}$ which only needs the mean and the variance of the RV X_l, and hence, this technique bypasses the need for direct integration of (4) and (5).

$$E[\sin(n\omega X_l)] = \frac{\sqrt{\pi}}{\Gamma(m_l)} G_{1,2}^{1,1}\left[\frac{n^2\omega^2\overline{\Omega}_l}{4m_l}\middle|\begin{matrix}1-m_l\\ \frac{1}{2};0\end{matrix}\right] = \frac{\sqrt{\pi}}{\Gamma(m_l)}\left(\frac{n^2\omega^2\overline{\Omega}_l}{4m_l}\right)^{\frac{1}{2}} G_{1,2}^{1,1}\left[\frac{n^2\omega^2\overline{\Omega}_l}{4m_l}\middle|\begin{matrix}\frac{1}{2}-m_l\\ 0;-\frac{1}{2}\end{matrix}\right]$$

$$= \frac{\sqrt{\pi}}{\Gamma(m_l)}\left(\frac{n^2\omega^2\overline{\Omega}_l}{4m_l}\right)^{\frac{1}{2}} \frac{\Gamma(m_l+\frac{1}{2})}{\Gamma(\frac{3}{2})}\, {}_1F_1\left(m_l+\frac{1}{2};\frac{3}{2};-\frac{n^2\omega^2\overline{\Omega}_l}{4m_l}\right) \tag{22}$$

$$= \sqrt{\frac{\overline{\Omega}_l}{m_l}}\,\frac{\Gamma\left(m_l+\frac{1}{2}\right)}{\Gamma(m_l)}\, n\omega\, {}_1F\left(m_l+\frac{1}{2};\frac{3}{2};-\frac{n^2\omega^2\overline{\Omega}_l}{4m_l}\right)$$

III. THE PROPOSED APPROXIMATION METHOD

The approximation method proposed in this paper is similar to that of [16] used in deriving bit-error probability of a spread spectrum multiple-access system. If $g(x)$ is a function of RV x, then by Taylor series expansion of $g(x)$, we have:

$$g(x) = g(\mu) + (x-\mu)\frac{\partial g(x)}{\partial x}\bigg|_{x=\mu} + \frac{1}{2}(x-\mu)^2\frac{\partial^2 g(x)}{\partial x^2}\bigg|_{x=\mu} + \ldots \tag{23}$$

where μ denotes the mean of the RV x. By a reasonable and well-known approximation method, $E\{g(x)\}$ can be approximated as:

$$E\{g(x\} \approx g(\mu) + \frac{1}{2}\sigma^2\frac{\partial^2 g(x)}{\partial^2}\bigg|_{x=\mu} = g(\mu) + \frac{1}{2}\sigma^2\frac{g(\mu+h)-2g(\mu)+g(\mu-h)}{h^2} \tag{24}$$

where σ is the standard deviation of RV x and h is the differential parameter. According to the result of [16], a proper approximate value for h equals $\sqrt{3}\sigma$ from accuracy point of view. Then the equation (24) can be written as:

$$E\{g(x)\} \approx \frac{2}{3}g(\mu) + \frac{1}{6}g\left(\mu+\sqrt{3}\sigma\right) + \frac{1}{6}g\left(\mu-\sqrt{3}\sigma\right) \tag{25}$$

If (25) is used, $E\{\cos(n\omega X_l)\}$ and $E\{\sin(n\omega X_l)\}$ can be described as are shown in (26) and (27) in the bottom of this page, where μ_{X_l} and σ_{X_l} are the mean and the standard deviation of RV X_l. As was shown in (26) and (27), the proposed approximation method provides a simple closed-form expressions for $E\{\cos(n\omega X_l)\}$ and $E\{\sin(n\omega X_l)\}$ which let us to formulate the required coefficients A_n and τ_n in (2) and (3) in closed-form analytical expressions. By applying (26) and (2) in (2) and (3), we have:

$$A_n = \prod_{l=1}^{L}\sqrt{\frac{4}{9}+\frac{4}{9}\cos(\sqrt{3}n\omega\sigma_{x_l})+\frac{1}{9}\cos^2(\sqrt{3}n\omega\sigma_{x_l})} \tag{28}$$

$$\tau_n = \sum_{l=1}^{L}\tan^{-1}\left(\frac{\frac{2}{3}\cos(n\omega\mu_{x_l})+\frac{1}{3}\cos(n\omega\mu_{x_l})\cos(\sqrt{3}n\omega\sigma_{x_l})}{\frac{2}{3}\sin(n\omega\mu_{x_l})+\frac{1}{3}\sin(n\omega\mu_{x_l})\cos(\sqrt{3}n\omega\sigma_{x_l})}\right) \tag{29}$$

Fig. 1 shows the approximated values of A_n and τ_n for n =1 to n = 40 (n is odd) for T = 100 and 200 for L i.i.d GG distributed RV's with $v_l = 1$, $m_l = 6$ for different values of L, based on our proposed approximation method in comparison with the exact results [10]. Results show that our proposed method can approximate A_n and τ_n with very good accuracy. In Fig. 2 the exact and approximated values of the CDF of a sum of L=3, 6, 8, 16 GG RVs with $m_l = m = 1$ and $v_l = v = 3$, $\overline{\Omega}_l = \Omega = 2$ dB and T=400 is shown. Results show that the developed approximation method estimates the values of CDF

$$E[\cos(n\omega X_l)] \approx \frac{2}{3}\cos(n\omega\mu_{X_l}) + \frac{1}{6}\cos(n\omega(\mu_{X_l} + \sqrt{3}\sigma_{X_l})) + \frac{1}{6}\cos(n\omega(\mu_{X_l} - \sqrt{3}\sigma_{X_l})) \quad (26)$$

$$E[\sin(n\omega X_l)] \approx \frac{2}{3}\sin(n\omega\mu_{X_l}) + \frac{1}{6}\sin(n\omega(\mu_{X_l} + \sqrt{3}\sigma_{X_l})) + \frac{1}{6}\sin(n\omega(\mu_{X_l} - \sqrt{3}\sigma_{X_l})) \quad (27)$$

very good. Moreover, by taking the first derivative of CDF with respect to x, the corresponding PDF can be obtained. As an example, in Fig. 3 the exact and estimated PDF of the GG distributed RV's are given. Results show that the proposed method also approximates the PDF of the sum of independent RVs with good accuracy.

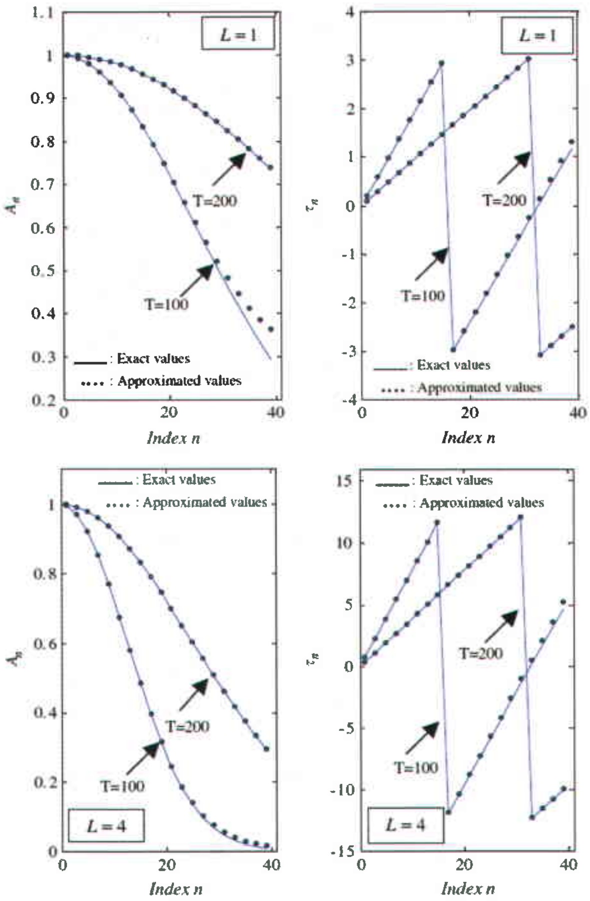

Figure 1. The values of An and □ n (for n=1–40, n is odd) based on our proposed approximation method and exact results for T =100

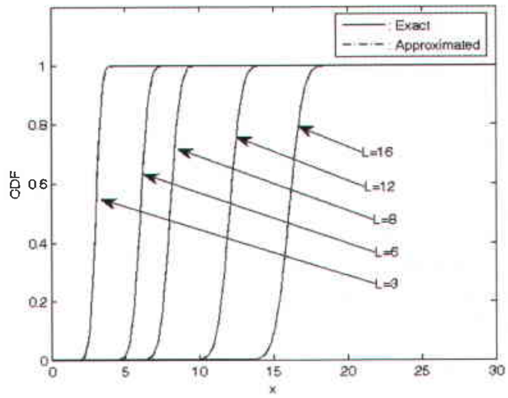

Figure 2. The exact and approximated values of the CDF of a sum of L GG RVs with $m_l = m = 1$ and $v_l = v = 3$, $\overline{\Omega}_l = \Omega = 2$ dB and T=400.

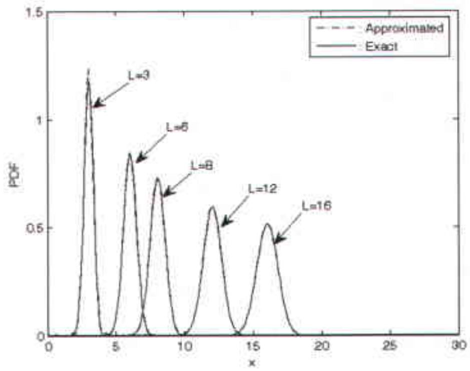

Figure 3. The exact and approximated values of the PDF of a sum of L GG RVs with $m_l = m = 1$, $v_l = v = 3$, $\overline{\Omega}_l = \Omega = 2$ dB and T=400.

IV. ACCURACY ANALYSI

The proposed approximation method is based on the Taylor series expansion over the expected function $E\{g(x)\}$ around the value of $g(\mu)$, where μ is the mean value of RV x, so the accuracy of this series improves when the concentration of distribution around its mean increases. As an example, in Fig. 4 the behavior of GG distribution with $v = 1$ for different values of m is shown. Fig.4 shows that in this case and with the increase of m, the concentration of the distribution around its mean increases and we expect that the accuracy of our proposed method improves. This fact is verified in Fig. 5.

Also the accuracy of the proposed method can be controlled by convergent series parameter T via parameter $\omega = 2\pi / T$. In particular, choosing large values of T will improve the accuracy and at the same time, can compensate the deleterious effects of large values of index n on the accuracy. In Fig. 10, the values of $E\{\cos(n\omega X_l)\}$ and $E\{\sin(n\omega X_l)\}$ are presented for GG distributed RV with $v_l = 1$ and $m_l = 1$ for different values of T. The results show that the accuracy improves as the parameter T increases. As was mentioned in [2], [9] and [10], the parameter T controls the accuracy of the CDF series and greater accuracy may be obtained using a larger value of T.

For example in [9] the value of T was chosen to lie between 200 and 500 corresponding to an estimated accuracy on the order of $\pm 10^{-10}$. Therefore, choosing larger value of T makes it possible to apply the developed approximation method for computation of the individual coefficient A_n and τ_n (for different values of n) in the CDF series with greater accuracy. Fig.11 shows the the exact and approximated values of A_n and θ_n for L =2 i.i.d GG distributed RVs with $v_l = v = 1$, $m_l = m = 2$, $\overline{\Omega}_l = \Omega = 10$ dB and $T = 200$.

V. Practical Application and Numerical Results

Presented numerical examples in previous sections verify the good accuracy of the proposed method in the approximation of the CDF and the PDF of the sum of independent RVs. In this section, to give a practical application, the proposed approximation method is used to analyze the performance of a multi-branch EGC receiver. Consider an EGC diversity combining system with L input branches operating over independent but not necessarily identically distributed GG fading channel. The baseband received signal at the l-th branch can be written as:

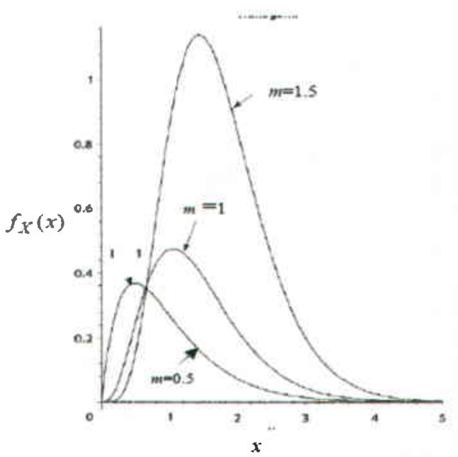

Figure 4. beheivior of GG distribution for $v = 1$ and different values of m

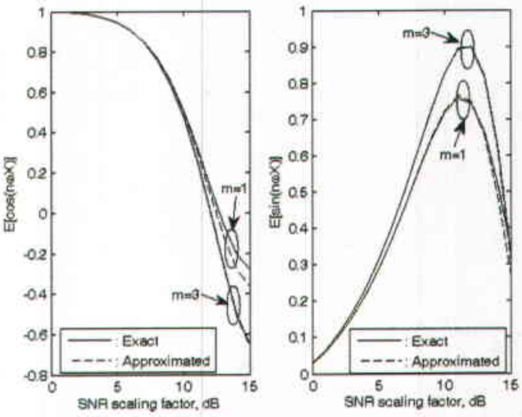

Figure 5. The exact and approximated values of $E\{\cos(n\omega X_l)\}$ and $E\{\sin(n\omega X_l)\}$ for GG distribution with $v_l = 1$, $n = 1$, T=200

$$y_l = w X_l \exp(j\varphi_l) + n_l, \quad l \in \{1,2,...,L\} \quad (28)$$

where X_l is the instantaneous fading envelope modeled as a GG distributed RV, w is the complex transmitted symbol with $E_s = E\left[|w|^2\right]$ which is the transmitted average symbols' energy, φ_l is the instantaneous phase of the channel which is assumed known for the receiver, and n_l is the instantaneous additive white Gaussian noise (AWGN) sample with single-sided power spectral density N_0 that is identical for all channels. The instantaneous SNR per symbol of the lth diversity branch can be described as:

$$\gamma_l = X_l^2 \frac{E_s}{N_0} \quad (29)$$

whose corresponding average SNR is described as follows:

$$\overline{\gamma_l} = E[X_l^2]\frac{E_s}{N_0} = \left(\frac{\overline{\Omega_l}}{m_l}\right)^{1/v_l} \frac{\Gamma(m_l + 1/v_l)}{\Gamma(m_l)} \quad (30)$$

The instantaneous EGC output SNR per symbol can be expressed as:

$$\gamma_{EGC} = (X_1 + X_2 + ... + X_L)^2 \frac{E_s}{LN_0} = X^2 \frac{E_s}{LN_0} \quad (31)$$

By using (2), the CDF of the EGC output SNR $\left(F_{\gamma_{EGC}}(\gamma)\right)$ can be calculated as follows:

$$P(\gamma_{EGC} \le \gamma) = P\left(X^2 \frac{E_s}{LN_0} \le \gamma\right) = P\left(X \le \sqrt{\frac{LN_0}{E_s}\gamma}\right) \quad (32)$$

Therefore,

$$F_{\gamma_{EGC}}(\gamma) = F_X\left(\sqrt{\frac{LN_0}{E_s}\gamma}\right) \qquad (33)$$

Where $F_X(x)$ was given by (1). If γ_{th} is a certain specified threshold, then the outage probability (P_{out}) is defined as the probability that γ_{EGC} falls below γ_{th}. The P_{out} can be obtained by replacing γ with γ_{th} in (33) as follows:

$$P_{out}(\gamma_{th}) = F_{\gamma_{EGC}}(\gamma_{th}) \qquad (34)$$

In Fig. 6, the values of P_{out} is calculated as a function of the normalized outage threshold, $\gamma_{th}/\overline{\gamma}$ for an L-branch EGC receiver over i.i.d GG fading channel with L=4, $v_l = v = 1.25$, $m_l = m = 2$. For comparison purpose, the curves for the corresponding upper bound obtained from [8] and the corresponding exact P_{out} obtained via computer simulation are included in the same figure. Comparing the approximated results evaluated based on the developed approximation method with the computer simulated ones and also with the upper bound of [8] demonstrates the accuracy of the proposed approximation method. In Fig. 7, the values of P_{out} is calculated as a function of the normalized outage threshold, $\gamma_{th}/\overline{\gamma}$ for an L-branch EGC receiver over i.i.d GG fading channel with $v_l = v = 1.5$, $m_l = m = 2$ and different values of L. Results show that P_{out} improves with an increase of L and also with the increase of the parameter v .In Fig. 13, P_{out} is plotted as function of the normalized outage threshold for L=8, $v_l = v = 1.5$ and different values of m_l . Results show that the performance improves with the increase of m.

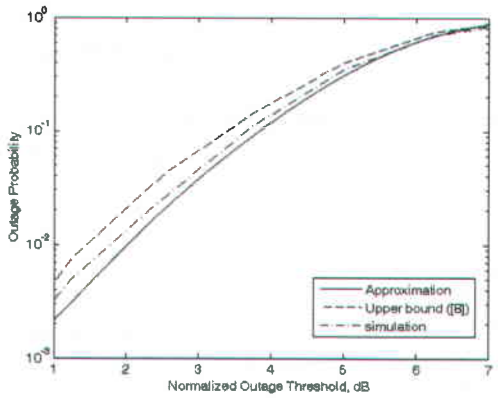

Figure 6. Outage probability of L branch EGC receiver for L=4 over i.i.d GG fading channel with $v_l = v = 1.25$ and $m_l = m = 2$

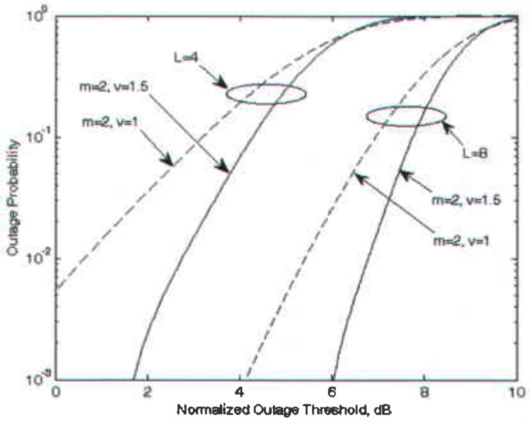

Figure 7. Outage probability of L branch EGC receiver for L=3, 8 over i.i.d GG fading channel derived.

Another important performance criteria is the amount of fading (AoF), which is a unified measure of the severity of fading [1]. AoF is defined as the ratio of the variance to the square average SNR per symbol as follows:

$$AoF = Var(\gamma_{EGC}^2)/\overline{\gamma_{EGC}}^2 \qquad (35)$$

The average and the variance of the output SNR can be calculated by using its PDF. Having numerically evaluated the required parameters using our proposed approximation method to calculate the PDF of SNR, in Fig. 16 the Amount of fading at the output of the EGC receiver is plotted as a function of v for L=8 and i.i.d branch SNR. As can be deduced from Fig. 16, AoF decreases as v and/or m increases.

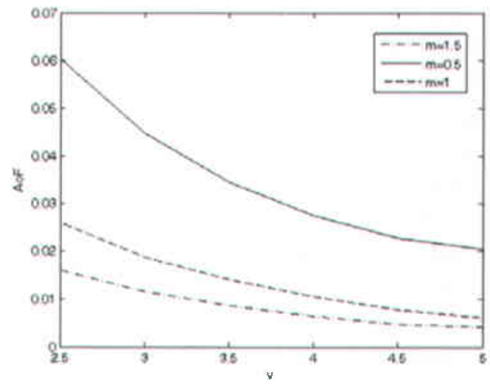

Figure 8. Amount of fading at the output of the EGC receiver as a function of v for L=8 and i.i.d branch SNRs.

VI. Conclusion

In this paper, the problem of finding the CDF of the sum of independent RVs is considered and a novel, simple approximation method has developed based on the convergent infinite series approach. The proposed method generalizes the applicability of the convergent infinite series approach for different distributions. As an example, in this paper, it has been used to calculate the statistical parameters of the GG distribution, which is a versatile envelope distribution that generalizes many of the commonly used models for multipath and shadow fading. The proposed method has been applied in the performance analysis of EGC receiver over GG fading channel. In a similar way, the developed method can be used to approximate the characteristic function of the fading envelope and efficiently used in the performance analysis of EGC receivers using characteristic function based approach over different independent fading channels.

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