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A Distributed Optimization Approach for Multi-Agent Systems over Delaying Networks

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Abstract—**This paper investigates a novel method to solve distributed optimization problems in the presence of communication delays between the networked agents that cooperate together to find an optimal solution of a global cost function composed of local ones. In the problem of distributed optimization in a network of multi-agent because of existing phenomena such as communication delay, deriving approaches having appropriate performance so that the states of all agents converge to the same value always has been a substantial challenge. Delay-dependent conditions in the form of linear matrix inequities are derived to analyze the convergence of the introduced scheme to the optimal solution. It is demonstrated that the maximum allowable time delay in the network and convergence rate of the optimization procedure are increased by the suggested strategy. Finally, comparative simulation results are considered to illustrate the superior performance of the introduced scheme compared to a rival one in the literature.**

Keywords: Distributed optimization; Communication delay; Linear Matrix Inequality; Multi-agent systems.

Article type: Research Article

I. Introduction

In the problem of distributed optimization, a global cost function composed of several local ones is optimized by the cooperation of networked agents. Each agent which has access only to one local objective function contributes to the solution of the problem through local computations and information exchange with the neighbors. Due to widespread applications of distributed optimization in real-world systems such as distributed estimation in sensor networks, motion planning, distributed predictive control, resource allocation over networks, and the economic power dispatch problem, this issue has attracted much attention in recent years [1-6].

The majority of works in the area of distributed optimization for multi-agent systems are focused on discrete-time formulations [7-11]. However, because of

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the simple application of the Lyapunov stability theorem for convergence analysis and also potential applications in continuous-time physical systems, much attention has been recently concentrated on the study of continuous-time distributed optimization [12-16].

In most of the aforementioned publications, communication channels between agents were assumed to be ideal, for instance, see the work of Weng et al [17], while time delays are inevitable in data exchange between agents in the networks due to a finite bandwidth of the communication medium and limited speed of agents for calculating and sending their outputs [18, 19]. Also, in practical systems such as unmanned aerial vehicles (UAVs), when the information would be transmitted by an UAV to others occurring time delay is inevitable. Compensation of destructive effect of communication delays on the performance and stability of solution process is an important challenge that is addressed only by a few papers in the literature. Therefore, it is beneficial to propose the approaches that consider communication delay in multi agent systems [20].

Authors in [21] by utilizing the mirror descent method, derived a distributed algorithm to solve a distributed optimization problem, which phenomenon of time delays in a multi-agent system are considered and analyzed the effects of delay on convergence rate. In [22], based on the dual averaging notion, an algorithm was derived to solve distributed cooperative optimization problems subject to delayed sub-gradient data in a networked system with delay. In [23], the problem of distributed optimization in the presence of inter-agent communication delays was solved via a proportional-integral consensus algorithm in a passivity-based framework. It was proved that the transmission delays can be handled while ensuring the convergence property using scattering transformation. Authors have developed a distributed optimization approach for a continuous-time multi-agent system subject to communication delays in [24]. Sufficient conditions in the form of linear matrix inequities were presented for convergence to analysis. In the paper by Lin et al [25], distributed optimization problem was solved by implementing a sub-gradient projection algorithm for a networked system subject to communication delays and nonidentical constraints. Moreover, a distributed optimization problem of multi-agent systems with delayed sampled data is considered in [26], then based on Lyapunov theory and graph the convergence of all the agents to the optimal solution was proved.

In this paper, inspired by [24] and [25], a novel technique is developed to solve efficiently the distributed optimization problem in the presence of communication delays utilizing the sub-gradient projection idea. The key idea to improve the approaches of [24] and [25] is that, for each agent based on the communication graph, a weighted information of its neighbors in the gradient term is implemented to update the value of the state of each agent. The main contributions of the proposed approach can be highlighted as follows:

(1) convergence of the proposed algorithm is assured using the Lyapunov-Krasovskii stability argument;
(2) it is demonstrated that compared to the rival method in [24], the convergence rate is increased by the proposed strategy; moreover, convergence to the optimal solution is achieved for higher values of transmission delay.

The remainder of this paper proceeds as follows. In section 2, first, the necessary background materials are called form literature and then the problem of distributed optimization for the system with multi-agent is modeled. The proposed algorithm is given in Section 3, then, the convergence of the optimization process is analyzed. In Section 5, comparative simulation results are presented to verify the superiority of the suggested scheme compared to the rival method in the literature. Ultimately, the conclusion is given in 6.

II. Problem Statement and Preliminaries

Considering a network of Λ agents interacting over a directed graph $\Xi(V,E)$ that consists of nodes (agents) set, $V=\{1,\ldots,\Lambda\}$ and an edge set, $E\subseteq V\times V$. An edge from υ_i to υ_j, described by (υ_i,υ_j), denotes that υ_j can obtain data from υ_i; then υ_j is known a neighbor of υ_i. The set $\Lambda_i=\{j\,|\,(i,j)\in E\}$ denotes all neighbors of the agent i. The considered graph is assumed to be strongly connected; namely, for every pair of nodes, there is a directed path connecting them. The adjacency matrix of the graph $\Xi(V,E)$ is represented by A which is a $\Lambda\times\Lambda$ matrix, whose entries a_{ij} are given as:

$$a_{ij}=\begin{cases}1 & if\ i\neq j\ and\ (i,j)\in\varepsilon,\\ 0 & otherwise.\end{cases} \tag{1}$$

The out-degree and in-degree of a node υ_i in the graph are $\bar{b}^i_{out}=\sum_{j=1}^{n}a_{ij}$ and $\bar{b}^i_{in}=\sum_{j=1}^{n}a_{ji}$, respectively. The considered graph Ξ is supposed to be weighted-balanced; that is, $\bar{b}^i_{in}=\bar{b}^i_{out}$ for any agent $\upsilon_i\in V$ [27, 28]. The overall cost function of the minimization problem is considered as follows:

$$F(x)=\sum_{i=1}^{\Lambda}f_i(x) \tag{2}$$

wherein the agent i only accesses to its local cost, $f_i(x)$ which m_i -strongly is convex in x. That is, $\nabla f_i(z)-\nabla f_i(x)^2\leq l_i(z-x)^2(\nabla f(z)-\nabla f(x))$ and ∇f_i is l_i -Lipchitz over R^n; that is for all $x,\,z\in C\subset \mathrm{R}^{\bar{b}}$. The aim is to find variables $x=(x_1,x_2,\ldots,x_n)\in\mathrm{R}^n$ such that the general objective function $F(x)$ in (2) attains its minimal value. For minimization of (2), the agents, V in the network coordinate their decisions through a set of communication links. To formulate this fact, the coupling among agents is transformed to a set of constraints [29]. Therefore, the minimization of structure (2) is defined as:

$$\begin{aligned} &\text{minimize} \quad F(x)=\sum_{i=1}^{\Lambda} f_i(x) \\ &s.t. \quad x_i = x_j,\ i=\{1,\dots,\Lambda\},\ j\in\Lambda_i, \end{aligned} \tag{3}$$

where $x_i \in \mathrm{R}^n$ denotes the state of the agent i. The role of each agent i is to contribute to finding the optimal solution of the problem (3) via cooperation with other agents. Inspired by the distributed optimization techniques in literature such as dual-decomposition and consensus fashions [15, 30, 31] the following networked system is obtained to find the solution of the optimization problem.

$$\begin{aligned} \dot{z}_i &= \gamma \sum_{j=1}^{N} \alpha_{ij}\left[x_i - x_j\right] \\ \dot{x}_i &= -\alpha\nabla f_i(x_i) - z_i - \beta\sum_{j=1}^{N}\alpha_{ij}\left[x_i - x_j\right] \end{aligned} \tag{4}$$

where α, β and γ are positive weights and the weighting that agent i associates to the agent j is α_{ij}. Note that the system (4) inherits the properties of dual-decomposition and consensus methods in solving the distributed optimization problem; namely, the fast transient behavior of consensus and desirable convergence of dual-decomposition.

Since the agents exchange information together through a communication graph subject to time-delay, the information of neighbors are not immediately in hand to be used in (4), i.e. at step time t, the amount of $x_j(t-d(t))$ is available instead of $x_j(t)$, where $d(t)$ is assigned to demonstrate time-varying delay among agents i and j, which is bounded as $0<d(t)\le \bar{d}$, with $\dot{d}(t)<1$. In the next section, system (4) is refined to tackle this issue efficiently. Before proceeding, some useful facts which will be employed in the derivation of our results are recalled from the literature.

Lemma 1. [15]. Laplace matrix concerned with the graph Ξ is described as the following representation:

$$L=\left[\ell_{ij}\right]\in \mathrm{R}^{\Lambda\times\Lambda},\ \ell_{ij}=\begin{cases}\sum_{j\in\Lambda_i} a_{ij} & i=j\\ -a_{ij} & i\neq j\end{cases} \tag{5}$$

It is alternatively can be construed as $L=D_{out}-A$, with $D_{out}=diag\left\{\bar{b}^1_{out},\bar{b}^2_{out},\dots,\bar{b}^n_{out}\right\} \in \mathrm{R}^{n\times n}$. The Laplacian L of graph Ξ is a positive semi-definite matrix and for a connected graph, the Laplacian has a single zero eigenvalue and the corresponding eigenvector is a vector of ones, defined by $\{k1_n : k\in \mathrm{R}\}$, which $1_n \in \mathrm{R}^n$.

Lemma 2. [32]. For an undirected connected graph with symmetric L, there exist $\Lambda-1$ real eigenvalues in the open right half plane (RHP) $0=\lambda_1<\lambda_2\le\dots\le\lambda_\Lambda$. The matrix L can be diagonalized through an orthogonal transformation as the following:

$$L=\mathbf{RJR}^{\mathbf{T}}$$

where the diagonal matrix J is as

$$\mathbf{J}=\begin{bmatrix}0 & 0_{1\times(\Lambda-1)}\\ 0_{(\Lambda-1)\times 1} & Y\end{bmatrix}$$

in which Y has a $(\Lambda-1)\times(\Lambda-1)$ diagonal matrix containing the mentioned RHP eigenvalues of L and $\mathbf{R}$ is constructed using the right eigenvectors of L as $\mathbf{R}=[r_1,r_2,\dots,r_N]$, wherein $r_i,\ i\in\{1,2,\dots,\Lambda\}$ defining the right eigenvectors of L where $r_i^T r_i=1$, and $r_1=1_\Lambda/\sqrt{\Lambda}$. Lemma 3. [33]. For any function $x(t)\in L_2[-s,0]$ and any positive matrix Ψ, the following inequality holds

$$\int_{-t}^{0} x^T(t)\Psi x(t)dt \ge \frac{1}{t}\left(\int_{-s}^{0} x^T(t)dt\right)\Psi\left(\int_{-s}^{0} x(t)dt\right)$$

Lemma 4 [34]. Assume that we have $Q_1\in \mathrm{R}^{n_1\times n_1},\dots,Q_{\bar{b}}\in\mathrm{R}^{n_{\bar{b}}\times n_{\bar{b}}}$ been positive matrices. For all $a_j>0$ with $\sum_j^{\bar{b}}\alpha_j=1$ and for all $S_{ji}\in\mathrm{R}^{n_j\times n_i}$, $j=1,2,\dots,\bar{b},\ i=1,\dots,j-1$, where

$$\begin{bmatrix}Q_j & S_{ji}\\ * & Q_i\end{bmatrix}\ge 0$$

The following relation holds:

$$\sum_{j=1}^{\bar{b}}\frac{1}{a_j}e_j^TQ_je_j \ge \begin{bmatrix}e_1\\ e_2\\ \vdots\\ e_{\bar{b}}\end{bmatrix}^T\begin{bmatrix}Q_1 & S_{12} & \cdots & S_{1\bar{b}}\\ * & Q_2 & \cdots & S_{2\bar{b}}\\ * & * & \ddots & \vdots\\ * & * & \cdots & Q_{\bar{b}}\end{bmatrix}\begin{bmatrix}e_1\\ e_2\\ \vdots\\ e_{\bar{b}}\end{bmatrix}$$

for all $e_1\in\mathrm{R}^{n_1},\dots,e_{\bar{b}}\in\mathrm{R}^{n_{\bar{b}}}$.

III. MAIN RESULTS

In this section, a novel continuous-time system is provided to be replaced with (4) in the case of delayed information; then convergence of the proposed method is analyzed. The merits of the proposed method are illustrated in the next section by simulation. The delayed subgradient information received at time step t is used to update evolution of $x_i(t)$ in the agent i can be defined as the following form:

$$\begin{aligned} \dot{z}_i(t) &= \gamma\sum_{j=1}^{\Lambda}\alpha_{ij}\left[x_i(t-d(t)) - x_j(t-d(t))\right] \\ \dot{x}_i(t) &= -\alpha\nabla f_i\left(w_a^i x_i(t) + \sum_{j\neq i, j\in\Lambda_i}^{\Lambda} w_n^{ij}x_j(t-d(t))\right) \\ &\quad - z_i(t) - \beta\sum_{j=1}^{\Lambda}\alpha_{ij}\left[x_i(t-d(t)) - x_j(t-d(t))\right] \end{aligned} \tag{6}$$

for i th agent and its neighbors, the weight coefficients w_a^i and w_n^{ij} are devoted, respectively; such that matrices $w_a = diag[w_a^i]_{\Lambda\times\Lambda}$ and $w_n=[w_n^{ij}]_{\Lambda\times\Lambda}$, $j\in\Lambda_i$, $j\neq i$ satisfy: $w_a^i+\sum_{j\neq i, j\in\Lambda_i}^{\Lambda} w_n^{ij}=1,\ i=1,\dots,\Lambda$. Now, we can rewrite system (6) in the compact form as the following representation:

$$
\begin{aligned}
\dot{\mathbf{z}}(t) &= \gamma \mathbf{L}\mathbf{x}(t-d(t)) \\
\dot{\mathbf{x}}(t) &= -\alpha \nabla F(\mathbf{w}_a \mathbf{x}(t) + \mathbf{w}_n \mathbf{x}(t-d(t))) - \mathbf{z}(t) \\
&\quad - \beta \mathbf{L}\mathbf{x}(t-d(t)),
\end{aligned} \tag{7}
$$

where $x = \left(x_1^T, x_2^T, ..., x_\Lambda^T\right)$, $\nabla F\left(x(t)\right) = (\nabla f_1(x_1(t))^T, \nabla f_2(x_2(t))^T, ..., \nabla f_\Lambda(x_\Lambda(t))^T)^T$, $z = \left(z_1^T, z_2^T, ..., z_\Lambda{}^T\right)$, $\mathbf{L} = L \otimes I_n \in \mathrm{R}^{\Lambda n \times \Lambda n}$ and $\otimes$ represents the Kronecker product. Note that time delay appears in the x component. Therefore, the initial value x needs to be known as a function x_t in $C\left(\left[-\bar{d}, 0\right], \mathrm{R}^{\Lambda n}\right)$ which is the space of continuous functions mapping the interval $\left[-\bar{d}, 0\right]$ into $\mathrm{R}^{\Lambda n}$ subject to norm $\varphi = \sup_{-\bar{d},, \vartheta,, 0} \varphi\left(\vartheta\right)$.

A. Optimality Analysis

In the following theorem, it is proven that the system (7) converges to the optimal solution of (3), subject to delayed transmission. Let $f_i, i \in V$ is differentiable and m_i -strongly convex, and its gradient is l_i - Lipschitz also, the graph Ξ is strongly connected and weighted-balanced.

Theorem 3.1: For any $\varepsilon \in \mathrm{R}^n$, $S\left(\varepsilon\right) = \{\left(z, x\right) | \left(1_\Lambda \otimes I_n\right)^T z = \varepsilon\}$ is a positive invariant set and x^* is an optimal solution of (3) if and only if $\left(-\alpha \nabla F\left(w_a x^* + w_n x^*\right), x^*\right) \in S\left(0_n\right)$ is an equilibrium of systems (7).

Proof. See the appendix.

B. Convergence analysis

Next, in order to find the optimal solution x^* to system (7), sufficient conditions in the form of LMIs are presented to guarantee the convergence.

Theorem 3.2: The System (7) with time-delay $\dot{d}\left(t\right) \le \varpi < 1$ and the initial condition $S\left(0_n\right)$ is convergent to the optimal solution of (3) if there exist $\left(\Lambda - 1\right) \times \left(\Lambda - 1\right)$ matrix S, P_4, P_2, and positive definite matrices Q_1, Q_2, Q_3, P_3 and positive scalars δ, ρ_{11} and ρ_{22} such that:

$$
\begin{bmatrix} Q_2 & S \\ * & Q_2 \end{bmatrix} > 0
$$

$$
\begin{pmatrix} \rho_{11} I_{\Lambda-1} & P_4 & \\ P_4^T & \rho_{22} I_{\Lambda-1} & P_2 \\ & P_2^T & P_3 \end{pmatrix} > 0
$$

$$
\begin{bmatrix}
\Pi_{11} & 0 & -P_4 & \Pi_{14} & \Pi_{15} & 0 & -\alpha P_4 & 0 & 0 \\
* & \Pi_{22} & -p_{22} I_{\Lambda-1} & \Pi_{24} & \Pi_{25} & S & \Pi_{27} & -\alpha P_4^T & 0 \\
* & * & \Pi_{33} & 0 & \Pi_{35} & 0 & -\alpha P_2 & 0 & -\bar{d} Q_2 \\
* & * & * & \Pi_{44} & \Pi_{45} & 0 & \Pi_{47} & \Pi_{48} & 0 \\
* & * & * & * & \Pi_{55} & \Pi_{56} & \Pi_{57} & \Pi_{58} & -\bar{d}\beta Q_2 J \\
* & * & * & * & * & * & 0 & 0 & 0 \\
* & * & * & * & * & * & -\delta I_{\Lambda-1} & 0 & -\bar{d}\alpha Q_2 \\
* & * & * & * & * & * & * & -\delta I_{\Lambda-1} & 0 \\
* & * & * & * & * & * & * & * & -Q_2
\end{bmatrix} < 0 \tag{8}
$$

With

$$\Pi_{22} = Q_1 - Q_2 + Q_3 + \underline{m}(2\gamma \hat{I} - (\frac{2\alpha}{a_1} p_{11})) a_1^2 I_{\Lambda-1},$$

$$\Pi_{11} = \underline{m}(2\gamma \hat{I} - (\frac{2\alpha}{a_1} p_{11})) a_1^2,$$

$$\Pi_{14} = \frac{1}{2} \underline{m}(2\gamma \hat{I} - (\frac{2\alpha}{a_1} p_{11})) z_{11},$$

$$\Pi_{15} = \frac{1}{2} \underline{m}(2\gamma \hat{I} - (\frac{2\alpha}{a_1} p_{11})) z_{12},$$

$$\Pi_{24} = \frac{1}{2} \underline{m}(2\gamma \hat{I} - (\frac{2\alpha}{a_1} p_{11})) z_{21},$$

$$
\begin{aligned}
\Pi_{25} &= \gamma P_2 J - \beta p_{22} J + Q_2 - S \\
&\quad + \frac{1}{2} \underline{m}(2\gamma \hat{I} - (\frac{2\alpha}{a_1} p_{11})) z_{22},
\end{aligned}
$$

$$\Pi_{27} = -\alpha(p_{22} - p_{11}) I_{\Lambda-1}, \Pi_{33} = -P_2^T - P_2,$$

$$\Pi_{35} = -\beta P_2 J + \gamma P_3 J,$$

$$\Pi_{44} = \underline{m}(2\gamma \hat{I} - (\frac{2\alpha}{a_1} p_{11})) g_{11},$$

$$\Pi_{45} = \frac{1}{2} \underline{m}(2\gamma \hat{I} - (\frac{2\alpha}{a_1} p_{11})) z_{12},$$

$$\Pi_{47} = \frac{1}{2}(\frac{2\alpha}{a_1} p_{11}) k_{12}, \Pi_{48} = \frac{1}{2}(\frac{2\alpha}{a_1} p_{11}) k_{11}$$

$$
\begin{aligned}
\Pi_{55} &= -(1-\varpi) Q_3 - 2Q_2 + S^T \\
&\quad + S + \underline{m}(2\gamma \hat{I} - (\frac{2\alpha}{a_1} p_{11})) g_{22},
\end{aligned}
$$

$$\Pi_{56} = Q_2 - S, \Pi_{57} = \frac{1}{2}(\frac{2\alpha}{a_1} p_{11}) k_{22},$$

$$\Pi_{58} = \frac{1}{2}(\frac{2\alpha}{a_1} p_{11}) k_{21}, \Pi_{66} = -Q_1 - Q_2$$

which $m = \min\{m_1, m_2, ..., m_\Lambda\}$ that m_i -strongly is convex, $\hat{I} = \max\{I_1, I_2, ..., I_\Lambda\}$ that I_i is Lipschitz coefficient, $a_1 \in diag[w_a^i]_{\Lambda\times N}$,

$$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = \mathbf{w}_a^T \mathbf{w}_n, \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \mathbf{w}_n^T \mathbf{w}_n$$

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = \mathbf{w}_n^T.$$

Proof: see the appendix.

It is noteworthy to mention that as stated in the next section, the values of the weight coefficients have been employed to derive sufficient conditions of the convergence. In other words, $\mathbf{w}_a$ and $\mathbf{w}_n$ affect the feasibility of LMIs (8).

IV. BENEFIT AND APPLICATION OF THIS KIND OF OPTIMIZATION

BENEFIT: This paper is assigned to investigate the problem of distributed multi-agent optimization over delaying networks. The optimization equation (6) is derived for objective enduring higher values of transmission delay and increasing of the converge rate. Differently from (4) in which the gradient of f_i is computed only on x_i, in the proposed system (6); the gradient term is evaluated on a weighted sum of x_i and all delayed x_j. In equation (6), both weight coefficients w_a and w_n would be chosen by the designer, and as demonstrated in the proceeding, these weight coefficients are implemented in establishing the sufficient conditions to assure the convergence of the proposed approach (i.e. in Linear Matrix Inequalities (LMIs) (8)). In addition, as shown later in the simulations, this idea leads to improved performance compared to [24].

APPLICATIONS: As we know, the distributed optimization technique is the backbone for the learning and formation control of many practical applications. For instance, networked mobile robots are collaborating with each other in order to reach a certain task, therefore, they have to cooperate together to minimize a collective cost such that a central computing station was not employed. As a case study, we can use this proposed optimization approach in [35], where networked mobile robots are connected through a communication graph and are aimed to minimize their speeds and maximize their distance from each other. This kind of optimization approach also could be implemented in problems such as [15], and collaborative multi-robot systems for search and rescue [36].

V. SIMULATION RESULTS

In this section, the advantages of the addressed distributed optimization method are provided by two examples.

Example 1: In this example, the results of the performance of the proposed scheme are compared to a rival one in the literature. A network including five agents is considered subject to a graph which can be seen in Figure. 1. The communication graph is weight-balanced and strongly connected.

Note that in order to easily see, fewer agents are deliberately considered in the network for simulation results. Although if we consider a network with more agents, better results would be obtained for the proposed method. Local cost functions with $x \in \mathrm{R}$ are as the following

$$f_1(x) = 0.9(x^2 + 2x + 1),\ f_2(x) = (x-4)^2,$$
$$f_3(x) = 0.5x^2 - 1,$$
$$f_4(x) = \sin\left(\frac{x}{2}\right) + \frac{x^2}{2},\ f_5(x) = (x-3)^2 \qquad (9)$$

which are strongly convex and globally Lipschitz. It can be simply verified that the general cost function $F = \sum_{i=1}^{\Lambda=5} f_i$, gets its minimum value $F_{min} = 16.05$, at $x_{min} = 1.51$. In the first simulation, the delay value is supposed to be constant $d = 0.62$. The values of the weight coefficients are chosen as follows:

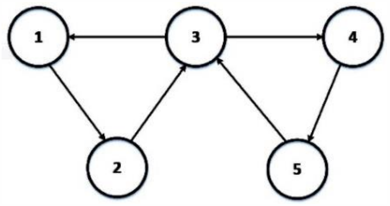

Fig 1. The communication topology of five agents

$$\mathbf{w}_a = \begin{bmatrix} 1.3 & 0 & 0 & 0 & 0 \\ 0 & 1.3 & 0 & 0 & 0 \\ 0 & 0 & 1.3 & 0 & 0 \\ 0 & 0 & 0 & 1.3 & 0 \\ 0 & 0 & 0 & 0 & 1.3 \end{bmatrix},$$

$$\mathbf{w}_n = \begin{bmatrix} 0 & 0 & -0.3 & 0 & 0 \\ -0.3 & 0 & 0 & 0 & 0 \\ 0 & -0.1 & 0 & 0 & -0.2 \\ 0 & 0 & -0.3 & 0 & 0 \\ 0 & 0 & 0 & -0.3 & 0 \end{bmatrix},$$

Figure 2 (**a**) shows the result obtained by the algorithm of [26] and Figure 2(**b**) depicts the consequence of the proposed algorithm with $\alpha = \beta = \gamma = 0.6$. Although, two methods can tolerate the same time delay it is clear that the convergence speed in Figure 2(**b**) is considerably faster than Figure 2(**a**).

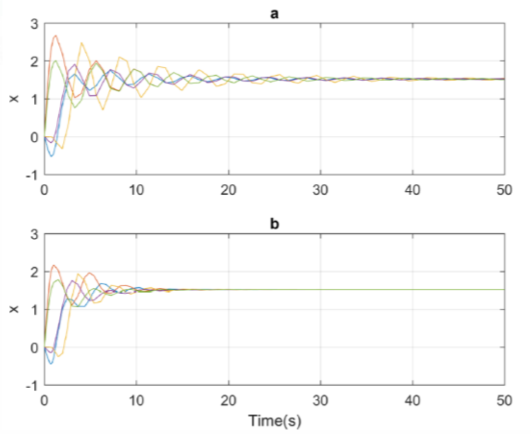

Fig 2. The agents' state trajectories for $d = 0.62$. (**a**): the method of [26] and (**b**) the proposed method.

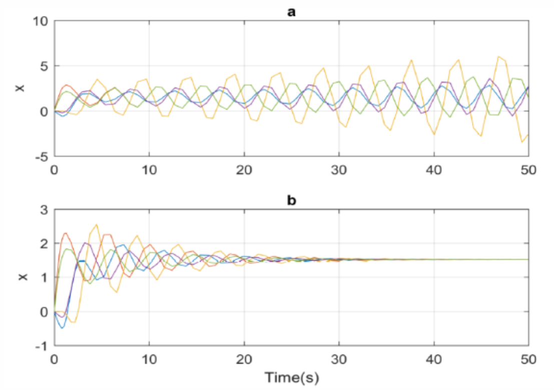

Fig 3. The agents' state trajectories for $d = 0.78$. (**a**): the method of [26] and (**b**) the proposed method.

In Figure. 3 state trajectories of agents are shown for $d = 0.78$ with rival and suggested schemes to verify that the maximum allowable delay in our method is much larger compared to [24]. To convenient evolution of maximum tolerable delay value, maximum allowable delays are reported in table 1 for the introduced method and rival schemes. It is clear that the converging criterion of the proposed method is less conservative compared to the maximum allowable delay of the rival scheme.

TABLE I. THE PERFORMANCE OF THE PROPOSED APPROACH

Approaches	*	**
The proposed method	0.82	0.8171
The Method of [24]	0.64	0.6093
* Actual value		
**Obtained from stability condition		

Example 2: Consider the following local cost functions subject to the communication graph in Figure. 4:

$$F(x) = \sum_{i=1}^{8} f_i(x). \tag{11}$$

where

$$\begin{aligned} &f_1(x) = (x-6)^2, f_2(x) = 0.8x^2 + 2, \\ &f_3(x) = 0.9\left(x^2 + 2x + 1\right), \\ &f_4(x) = (x-4)^2, f_5(x) = 0.5x^2 - 1, \\ &f_6(x) = \sin\left(\frac{x}{2}\right) + \frac{x^2}{2}, \\ &f_7(x) = 0.6x^2 + x, f_8(x) = (x-3)^2 \end{aligned} \tag{12}$$

Now, System (7) including eight agents is employed to solve optimization structure (11) subject to (12). It is assumed that we have the communication delays via a determined value. The parameters in system (7) are selected as $\alpha = 0.6, \beta = \gamma = 0.5$. The values of the weight coefficients are chosen as follows:

$$\mathbf{w}_a = \begin{bmatrix} 1.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.3 \end{bmatrix},$$

$$\mathbf{w}_n = \begin{bmatrix} 0 & -0.3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & 0 & 0 & -0.1 & -0.1 & 0 \\ -0.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.3 \\ 0 & -0.3 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

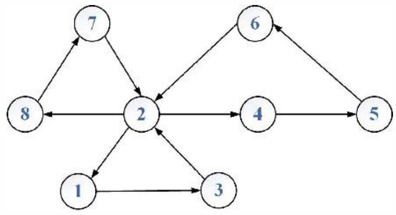

Fig 4. The communication topology of eight agents.

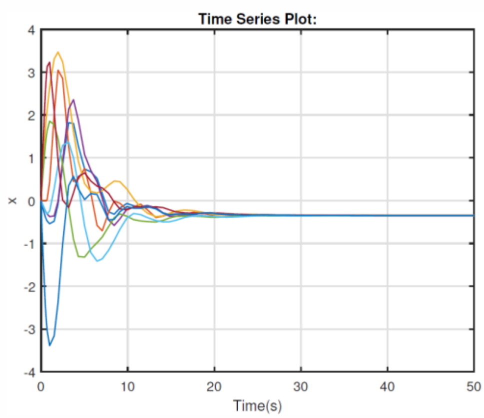

Fig 5. The agents' state trajectories for $d = 0.6$.

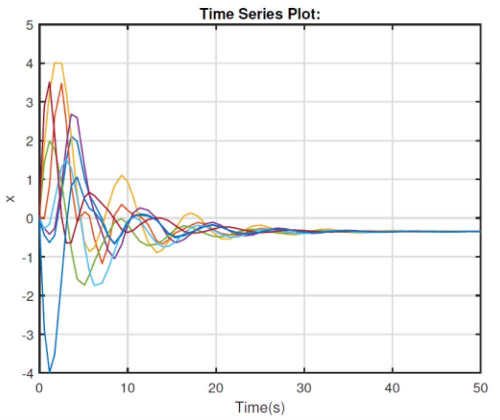

Fig 6. The agents' state trajectories for $d = 0.72$.

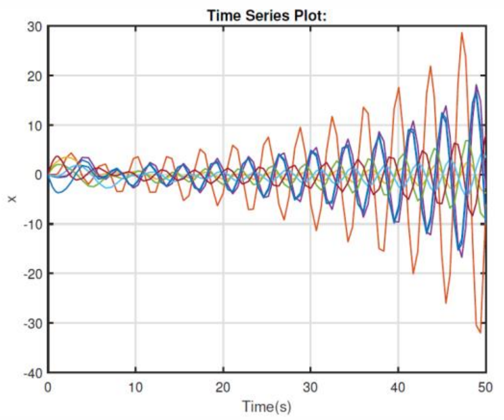

Fig 7. The agents' state trajectories for $d = 0.80$ .of sensor design.

According to the communication graph in Figure. 4, the simulation results for time-delay $d = 0.60$, $d = 0.72$ and $d = 0.80$ are shown in Figures. 5-7, respectively. Figure. 7, unlike Figure. 5 and 6, $x(t)$ is not convergent.

VI. Conclusions

In this paper, the problem of distributed multi-agent optimization over delaying networks is investigated. A novel method has been presented to solve a distributed continuous-time optimization problem with a multi-agent system subject to communication delay. Sufficient conditions have been derived in terms of LMIs to check the convergence of the algorithm to the optimal solution utilizing Lyapunov-Krasovskii's theory. Comparative simulation results have been presented to demonstrate that maximum allowable delay and rate of convergence are both improved compared to the recent rival method in the literature. Many issues are still open for future research, for instance considering uncertainties, and quantization effects [37] in the parameters of a multi-agent system.

VII. Appendix

Proof. of Theorem 3.1.

Regarding that communication delay occurs in the x component of the delayed system (7), the initial value of system (7) is $\mathbf{z}(0)$. From Lemma 2, we have $(1_\Lambda \otimes I_\Lambda)^T (L \otimes I_n) = (0_\Lambda \otimes I_\Lambda)^T$; So

$$(1_\Lambda \otimes I_\Lambda)^T \dot{\mathbf{z}}(t) = \gamma (1_\Lambda \otimes I_\Lambda)^T L\mathbf{x}(t - d(t)) \equiv 0_n \tag{13}$$

which states that the set $S(\varepsilon)$ is invariant for any ε, therefore:

$$(1_\Lambda \otimes I_n)^T \mathbf{z}(t) = (1_\Lambda \otimes I_n)^T \mathbf{z}(0) \tag{14}$$

which implies $\varepsilon = (1_\Lambda \otimes I_\Lambda)^T \mathbf{z}(0)$.

Additionally, assume $(\mathbf{z}^*, \mathbf{x}^*)$ denotes the equilibrium point of (7) in $S(0_n)$, and recall the Laplace matrix L concerned with a graph, then from Lemma 2, we have

$$0 = \gamma \mathbf{L}\mathbf{X}^* 0 = -\alpha \nabla F(\mathbf{X}^*) - \beta \mathbf{L}\mathbf{X}^* - \mathbf{z}^* \tag{15}$$

wherein $\mathbf{X}^* = 1_\Lambda \otimes \mathbf{x}^*$. By multiplying $(1_\Lambda \otimes I_n)^T$ to (15) from the left and regarding (13), we have $-\alpha \nabla F(X^*) = 0$. Regarding weight coefficients assigned to agents in (6), the related $X^* = w_a x^* + w_n x^*$ results in $X^* = x^*$ and we have the following:

$$-\alpha \nabla F(X^*) = -\alpha \sum_{i=1}^{\Lambda} \nabla f^i(x^*) = \tag{16}$$

to find the optimal condition of X^*. In addition, due to $0 = \gamma L x^*$ and $0 = -\alpha \nabla F(\mathbf{X}^*) - \beta \mathbf{L}\mathbf{x}^* - \mathbf{z}^*$, we can obtain $\mathbf{z}^* = -\alpha \nabla F(\mathbf{X}^*)$. The proof of converse is uncomplicated. Note that the initial value for solving the problem (6) requires to be located in $S(0_n)$ otherwise, if $\varepsilon = (1_\Lambda \otimes I_\Lambda)^T \mathbf{z}(0) \neq 0_n$, due to (15) we have $\alpha (1_\Lambda \otimes I_\Lambda)^T \nabla F(\bar{x}) \neq 0_n$ that verifies the system (6) may converge to the vector $\bar{x}$ that is not the optimal solution.

Proof. of Theorem 3.2. Consider definitions $\tilde{\mathbf{z}} = \mathbf{z} - \mathbf{z}^*$ and $\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}^*$, the equilibrium point transformers to the origin by replacing $\tilde{\mathbf{z}}$ and $\tilde{\mathbf{x}}$ within the structure (7).

$$\begin{aligned} \tilde{\mathbf{z}} &= (\mathbf{R} \otimes I_n)\mathbf{v}, \\ \tilde{\mathbf{x}} &= (\mathbf{R} \otimes I_n)(\mathbf{w}_a \mathbf{u}(t)) + (\mathbf{R} \otimes I_n) \\ &\quad \times (\mathbf{w}_n \mathbf{u}(t - d(t))) \end{aligned} \tag{17}$$

$\mathbf{R}$ is in the form of $[r_1 \quad R_2]^T$ with $r_1 = (1/\sqrt{\Lambda}) 1_\Lambda$. Let $\mathbf{v}$ be $(v_1^T, v_{2:\Lambda}^T)^T$ with $v_1 \in \mathrm{R}^n$, $v_{2:\Lambda} \in \mathrm{R}^{(\Lambda-1)n}$ and in a similar process: $\mathbf{u} = (u_1^T, u_{2:\Lambda}^T)^T$. Since the initial condition belongs to $S(0_n)$, system (7) can be converted to

$$\dot{v}_1 = 0_n$$
$$\dot{v}_{2:\Lambda} = \gamma \mathbf{J} u_{2:\Lambda}\left(t-d\left(t\right)\right)$$
$$\dot{u}_1 = -\alpha \mathbf{r}_1^T h\left(\tilde{\mathbf{z}}\right)$$
$$\dot{u}_{2:\Lambda} = -\alpha \mathbf{R}_2^T h\left(\tilde{\mathbf{z}}\right) - \beta \mathbf{J} u_{2:\Lambda}\left(t-d\left(t\right)\right) - v_{2:\Lambda}\left(t\right) \quad (10)$$

wherein $\mathbf{J} = J \otimes I_n$, $\mathbf{r}_1 = r_1 \otimes I_n$, $\mathbf{R}_2 = R_2 \otimes I_n$ and

$$h\left(\tilde{\mathbf{z}}\right) = \nabla F\left(\tilde{\mathbf{z}} + \mathbf{z}^*\right) - \nabla F\left(\mathbf{z}^*\right) \quad (19)$$

Concerning the initial condition and $\dot{v}_1 = 0_n$, we can have $v_1 \equiv 0$. Then, we only need to focus on the other three relations of (18).

$$V = V_1 + V_2 + V_3 + V_4 \quad (20)$$

with

$$V_1 = \eta^T \mathbf{P} \eta V_2 = \int_{t-\bar{d}}^{t} u_{2:\Lambda}^T\left(s\right)\mathbf{Q}_1 u_{2:\Lambda}\left(s\right) ds V_3$$
$$= \bar{d}\int_{-\bar{d}}^{0}\int_{t+\theta}^{t} \dot{u}_{2:\Lambda}^T\left(s\right)\mathbf{Q}_2 \dot{u}_{2:\Lambda}\left(s\right) ds d\theta V_4$$
$$= \int_{t-d(t)}^{t} u_{2:\Lambda}^T\left(s\right)\mathbf{Q}_3 u_{2:\Lambda}\left(s\right) ds \quad (21)$$

In which $\mathbf{Q}_i = Q_i \otimes I_n \left(i = 1,2,3\right)$, $\eta = (u_1^T, u_{2:\Lambda}^T, v_{2:\Lambda}^T)^T$, and $\mathbf{P} = P \otimes I_n$ such that:

$$P = \begin{pmatrix} p_{11} & P_4 & \\ P_4^T & p_{22} I_{\Lambda-1} & P_2 \\ & P_2^T & P_3 \end{pmatrix} \quad (22)$$

It is obvious that $V > 0$. Derivation of V_1, V_2, V_3 and V_4 are calculated along the trajectories of systems (21) as the following representation:

$$\dot{V}_1 = -2\alpha p_{11} u_1^T(t)(\mathbf{r}_1^T h(\tilde{\mathbf{z}})) - 2\alpha p_{22} u_{2:\Lambda}^T(t)\mathbf{R}_2^T h(\tilde{\mathbf{z}})$$
$$-2\beta p_{22} u_{2:\Lambda}^T(t)\mathbf{J} u_{2:\Lambda}(t-d(t)) - 2\alpha u_1^T(t) P_4 \mathbf{R}_2^T h(\tilde{\mathbf{z}})$$
$$-2\beta u_1^T(t) P_4 \mathbf{J} u_{2:\Lambda}(t-d(t))$$
$$-2u_1^T(t) P_4 v_{2:\Lambda}(t) - 2\alpha u_{2:\Lambda}^T(t) P_4^T \mathbf{r}_1^T h(\tilde{\mathbf{z}})$$
$$-2p_{22} u_{2:\Lambda}^T(t) v_{2:\Lambda}(t)$$
$$+2\gamma u_{2:\Lambda}^T(t) P_2 \mathbf{J} u_{2:\Lambda}(t-d(t))$$
$$-2\alpha v_{2:\Lambda}^T(t) P_2 \mathbf{R}_2^T h(\tilde{\mathbf{z}})$$
$$-2\beta v_{2:\Lambda}^T(t) P_2 \mathbf{J} u_{2:\Lambda}(t-d(t)) \quad (23)$$
$$-2 v_{2:\Lambda}^T(t) P_2 v_{2:\Lambda}(t)$$
$$+2\gamma v_{2:\Lambda}^T(t) P_3 \mathbf{J} u_{2:\Lambda}(t-d(t)),$$

$$\dot{V}_2 = u_{2:\Lambda}^T\left(t\right)\mathbf{Q}_1 u_{2:\Lambda}\left(t\right) - u_{2:\Lambda}^T\left(t-\bar{d}\right)\mathbf{Q}_1 u_{2:\Lambda}\left(t-\bar{d}\right) \quad (24)$$

$$\dot{V}_3 = \bar{d}\int_{-\bar{d}}^{0} \dot{u}_{2:\Lambda}^T(s)\mathbf{Q}_2 \dot{u}_{2:\Lambda}(t) d\theta$$
$$-\bar{d}\int_{-\bar{d}}^{0} \dot{u}_{2:\Lambda}^T(t+\theta)\mathbf{Q}_2 \dot{u}_{2:\Lambda}(t+\theta) d\theta \quad (25)$$
$$= \bar{d}^2 \dot{u}_{2:\Lambda}^T(t)\mathbf{Q}_2 \dot{u}_{2:\Lambda}(t)$$
$$-\bar{d}\int_{t-\bar{d}}^{t} \dot{u}_{2:\Lambda}^T(\theta)\mathbf{Q}_2 \dot{u}_{2:\Lambda}(\theta) d\theta$$

$$\dot{V}_4 = u_{2:\Lambda}^T(t)\mathbf{Q}_3 u_{2:\Lambda}(t)$$
$$-(1-\dot{d}(t)) u_{2:\Lambda}^T(t-d(t))\mathbf{Q}_3 u_{2:\Lambda}(t-d(t)) \quad (26)$$
$$„ u_{2:\Lambda}^T(t)\mathbf{Q}_3 u_{2:\Lambda}(t) - (1-\varpi) u_{2:\Lambda}^T(t-d(t))$$
$$\times \mathbf{Q}_3 u_{2:\Lambda}(t-d(t)).$$

Now, consider the first two expressions in $\dot{V}_1$, the following relations can be obtained:

$$\Omega = -2\alpha p_{11} u_1^T(t)(\mathbf{r}_1^T h(\tilde{\mathbf{z}}))$$
$$-2\alpha p_{22} u_{2:\Lambda}^T(t)\mathbf{R}_2^T h(\tilde{\mathbf{z}}) \quad (27)$$
$$= -2\alpha p_{11} \mathbf{u}^T(t)\mathbf{R}^T h(\tilde{\mathbf{z}})$$
$$-2\alpha(p_{22} - p_{11}) u_{2:\Lambda}^T \mathbf{R}_2^T h(\tilde{\mathbf{z}})$$

Note that $\mathbf{R}^T\mathbf{R} = I_\Lambda$ yields to $\mathbf{R}_2^T\mathbf{R}_2 = I_{\Lambda-1}$, then we have $\mathbf{R}_2 = 1$. from the right term in equation (17), we have $\tilde{\mathbf{z}}^T = \mathbf{u}^T\left(t\right)\mathbf{w}_a^T\mathbf{R}^T + \mathbf{u}^T\left(t-d\right)\mathbf{w}_n^T\mathbf{R}^T$. On the other hand, the following inequality holds:

$$\mathbf{u}^T\mathbf{R}^T = \frac{1}{w_a^i}\mathbf{u}^T\mathbf{w}_a^T\mathbf{R}^T + \frac{1}{w_a^i}\mathbf{u}^T(t-d)\mathbf{w}_n^T\mathbf{R}^T - \frac{1}{w_a^i}\mathbf{u}^T(t-d)\mathbf{w}_n^T\mathbf{R}^T \quad (28)$$

Where $\mathbf{u}$ refers to $\mathbf{u}\left(t\right)$, replacing (28) in (27) and using $\frac{1}{w_a^i}\mathbf{w}_a = \frac{1}{w_a^i}\mathbf{w}_a^T = I_{\Lambda\times\Lambda}$ results in the equation:

$$\Omega = -\frac{2\alpha}{\alpha_1} p_{11}\tilde{\mathbf{z}}^T h(\tilde{\mathbf{z}}) + \frac{2\alpha}{\alpha_1} p_{11}\mathbf{u}^T(t-d)\mathbf{w}_n^T\mathbf{R}^T h(\tilde{\mathbf{z}}) - 2\alpha(p_{22} - p_{11}) u_{2:\Lambda}^T \mathbf{R}_2^T h(\tilde{\mathbf{z}}) \quad (29)$$

As regards F is strongly convex and ∇F is Lipchitz, the followings hold:

$$\mathbf{R}_2^T h\left(\tilde{\mathbf{z}}\right)^2 „ h\left(\tilde{\mathbf{z}}\right)^2 „ \hat{l}\tilde{\mathbf{z}}^T h\left(\tilde{\mathbf{z}}\right) \quad (30)$$
$$\tilde{\mathbf{z}}^T h\left(\tilde{\mathbf{z}}\right) \geq \underline{m}\tilde{\mathbf{z}}^T\tilde{\mathbf{z}} = \underline{m}\mathbf{u}^T\mathbf{u} \quad (31)$$

in which

$$\mathbf{u}^T\mathbf{u} = \left(\mathbf{u}^T\mathbf{w}_a{}^T\mathbf{R}^T + \mathbf{u}^T\left(t-d\right)\mathbf{w}_n^T\mathbf{R}^T\right)(\mathbf{R}\mathbf{w}_a\mathbf{u} + \mathbf{R}\mathbf{w}_n\mathbf{u}\left(t-d\right)) .$$

Inequality (31) yields to

$$\delta\hat{l}\tilde{\mathbf{z}}^T h\left(\tilde{\mathbf{z}}\right) - \delta(\mathbf{R}_2 h\left(\tilde{\mathbf{z}}\right))^T\left(\mathbf{R}_2 h\left(\tilde{\mathbf{z}}\right)\right)..0$$
$$\delta\hat{l}\tilde{\mathbf{z}}^T h\left(\tilde{\mathbf{z}}\right) - \delta(\mathbf{r}_1 h\left(\tilde{\mathbf{z}}\right))^T\left(\mathbf{r}_1 h\left(\tilde{\mathbf{z}}\right)\right)..0 \quad (32)$$

So, finally, we have:

$$\Omega \leq (2\delta\hat{l} - 2\frac{\alpha}{\alpha_1}p_{11})\tilde{\mathbf{z}}^T h(\tilde{\mathbf{z}})$$
$$+2\frac{\alpha}{\alpha_1}p_{11}\mathbf{u}^T(t-d)\times\mathbf{w}_n^T\mathbf{R}^T h(\tilde{\mathbf{z}})$$
$$-2\alpha(p_{22} - p_{11}) u_{2:\Lambda}^T \mathbf{R}_2^T h(\tilde{\mathbf{z}})$$
$$-\delta(\mathbf{R}_2 h(\tilde{\mathbf{z}}))^T(\mathbf{R}_2 h(\tilde{\mathbf{z}}))$$
$$-\delta(\mathbf{r}_1 h(\tilde{\mathbf{z}}))^T(\mathbf{r}_1 h(\tilde{\mathbf{z}})) \leq \underline{m}(\delta\hat{I} - 2\alpha p_{11})\mathbf{u}^T\mathbf{u} \quad (33)$$

$$+2\frac{\alpha}{\alpha_1}p_{11}\mathbf{u}^T(t-d)\times\mathbf{w}_n^T\mathbf{R}^T h(\tilde{\mathbf{z}})$$
$$-2\alpha(p_{22}-p_{11})u_{2:\Lambda}^T\mathbf{R}_2^T h(\tilde{\mathbf{z}})$$
$$-\delta(\mathbf{R}_2 h(\tilde{\mathbf{z}}))^T(\mathbf{R}_2 h(\tilde{\mathbf{z}}))-\delta(\mathbf{r}_1 h(\tilde{\mathbf{z}}))^T(\mathbf{r}_1 h(\tilde{\mathbf{z}})).$$

Note that we can have the following equation

$$\mathbf{u}^T\mathbf{u}=\mathbf{u}^T\mathbf{w}_a^T\mathbf{w}_a\mathbf{u}+\mathbf{u}^T\mathbf{w}_a^T\mathbf{w}_n\mathbf{u}(t-d)$$
$$+\mathbf{u}^T(t-d)\mathbf{w}_n^T\mathbf{w}_a\mathbf{u}+\mathbf{u}^T(t-d)\mathbf{w}_n^T\mathbf{w}_n\mathbf{u}(t-d)$$

and be defined $\mathbf{u}=(u_1^T,u_{2:\Lambda}^T)^T$ in terms of u_1^T and $u_{2:\Lambda}^T$. Since $\mathbf{w}_a^T\mathbf{w}_a=a_1^2 I_\Lambda$ we have:

$$\mathbf{u}^T\mathbf{w}_a^T\mathbf{w}_a\mathbf{u}=\begin{bmatrix}u_1^T & u_{2:\Lambda}^T\end{bmatrix}\mathbf{w}_a^T\mathbf{w}_a\begin{bmatrix}u_1^T\\ u_{2:\Lambda}^T\end{bmatrix} \quad (34)$$
$$=u_1^T a_1^2 u_1+u_{2:\Lambda}^T a_1^2 I_{\Lambda-1}u_{2:\Lambda}^T$$

Let $\mathbf{w}_a^T\mathbf{w}_n=\begin{bmatrix}z_{11} & z_{12}\\ z_{21} & z_{22}\end{bmatrix}$ then we can obtain

$$\mathbf{u}^T\mathbf{w}_a^T\mathbf{w}_n\mathbf{u}(t-d)=u_1^T z_{11}u_1$$
$$+u_{2:\Lambda}^T z_{22}u_{2:\Lambda}^T(t-d)+u_1^T z_{12}u_{2:\Lambda}^T(t-d) \quad (35)$$
$$+u_{2:\Lambda}^T(t-d)z_{21}u_1(t-d)$$

Defining $\mathbf{w}_n^T\mathbf{w}_a=\begin{bmatrix}Y_{11} & Y_{12}\\ Y_{21} & Y_{22}\end{bmatrix}$, we reach to

$$\mathbf{u}^T(t-d)\mathbf{w}_n^T\mathbf{w}_a\mathbf{u}=u_1^T(t-d)Y_{11}u_1(t-d)$$
$$+u_{2:\Lambda}^T(t-d)Y_{22}u_{2:\Lambda}+u_1^T(t-d)Y_{12}u_{2:\Lambda} \quad (36)$$
$$+u_{2:\Lambda}^T(t-d)Y_{21}u_1$$

and, defining $\mathbf{w}_n^T\mathbf{w}_n=\begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix}$, we have

$$\mathbf{u}^T(t-d)\mathbf{w}_n^T\mathbf{w}_n\mathbf{u}(t-d)=u_1^T(t-d)g_{11}u_1(t-d)$$
$$+u_{2:\Lambda}^T(t-d)g_{22}u_{2:\Lambda}^T(t-d)$$
$$+u_1^T(t-d)g_{12}u_{2:\Lambda}^T(t-d) \quad (37)$$
$$+u_{2:\Lambda}^T(t-d)g_{21}u_1(t-d)$$

by parsing $\mathbf{w}_n^T=\begin{bmatrix}k_{11} & k_{12}\\ k_{21} & k_{22}\end{bmatrix}$ yields to,

$$2\frac{\alpha p_{11}}{\alpha_1}\mathbf{u}^T(t-d)\mathbf{w}_n^T\mathbf{R}^T h(\tilde{\mathbf{z}})=$$
$$2\frac{\alpha p_{11}}{\alpha_1}u_1^T(t-d)k_{11}\mathbf{r}_1^T h(\tilde{\mathbf{z}})$$
$$+2\frac{\alpha p_{11}}{\alpha_1}u_{2:\Lambda}^T(t-d)k_{22}\mathbf{R}_2^T h(\tilde{\mathbf{z}})$$
$$+2\frac{\alpha p_{11}}{\alpha_1}u_1^T(t-d)k_{12}\mathbf{R}_2^T h(\tilde{\mathbf{z}})$$
$$+2\frac{\alpha p_{11}}{\alpha_1}u_{2:\Lambda}^T(t-d)k_{21}\mathbf{r}_1^T h(\tilde{\mathbf{z}})$$

Regarding (33)-(37), upper-bounded of Ω can be acquired that ultimately, it will be used to make final LMIs.

Additionally, by substituting $\dot{u}_{2:\Lambda}=-\alpha\mathbf{R}_2^T h(\tilde{\mathbf{z}})-\beta\mathbf{J}u_{2:\Lambda}(t-d(t))-v_{2:\Lambda}(t)$ into the first term of $\dot{V}_3$, we have:

$$\bar{d}^2\dot{u}_{2:\Lambda}^T(t)Q_2\dot{u}_{2:\Lambda}(t)=\tilde{\eta}^T(t)U^T Q_2^{-1}U\tilde{\eta}(t) \quad (38)$$

Where

$$\tilde{\eta}(t)=(u_1^T(t),u_{2:\Lambda}^T(t),v_{2:\Lambda}^T(t),$$
$$u_1^T(t-d(t))u_{2:\Lambda}^T(t-d(t)),u_{2:\Lambda}^T(t-\bar{d}), \quad (39)$$
$$h^T(\tilde{\mathbf{z}})\mathbf{R}_2,h^T(\tilde{\mathbf{z}})\mathbf{r}_1)^T$$

and

$$U=[0,0,-\bar{d}Q_2,0,-\bar{d}\beta Q_2\mathbf{J},0,-\bar{d}\alpha Q_2,0] \quad (40)$$

Now, consider the following term in (25):

$$-\bar{d}\int_{t-d(t)}^{t}\dot{u}_{2:\Lambda}^T(\theta)Q_2\dot{u}_{2:\Lambda}(\theta)d\theta$$

Then, by using Lemma 2 and Lemma 2, the following relations can be obtained:

$$-\bar{d}\int_{t-\bar{d}}^{t}\dot{u}_{2:\Lambda}^T(\theta)Q_2\dot{u}_{2:\Lambda}(\theta)d\theta=$$
$$-\bar{d}\int_{t-\bar{d}}^{t-d(t)}\dot{u}_{2:\Lambda}^T(\theta)Q_2\dot{u}_{2:\Lambda}(\theta)d\theta$$
$$-\bar{d}\int_{t-d(t)}^{t}\dot{u}_{2:\Lambda}^T(\theta)Q_2\dot{u}_{2:\Lambda}(\theta)d\theta,,\ -\frac{\bar{d}}{d(t)}e_1^T Q_2 e_1$$
$$-\frac{\bar{d}}{\bar{d}-d(t)}e_2^T Q_2 e_2=-\begin{pmatrix}e_1\\ e_2\end{pmatrix}^T$$
$$\begin{pmatrix}\frac{\bar{d}}{d(t)}Q_2 & \\ & \frac{\bar{d}}{\bar{d}-d(t)}Q_2\end{pmatrix}\begin{pmatrix}e_1\\ e_2\end{pmatrix},, \quad (41)$$
$$-\begin{pmatrix}e_1\\ e_2\end{pmatrix}^T\begin{pmatrix}Q_2 & S_{12}\\ * & Q_2\end{pmatrix}\begin{pmatrix}e_1\\ e_2\end{pmatrix}$$

where $e_1=u_{2:\Lambda}(t)-u_{2:\Lambda}(t-d(t))$ and $e_2=u_{2:\Lambda}(t-d(t))-u_{2:\Lambda}(t-\bar{d})$. combining (38) and (41), the following upper bound is obtained for $\dot{V}_3$

$$\dot{V}_3\le\tilde{\eta}^T(t)U^T\mathbf{Q}_2^{-1}U\tilde{\eta}(t)-\begin{pmatrix}e_1\\ e_2\end{pmatrix}^T\begin{pmatrix}Q_2 & S_{12}\\ * & Q_2\end{pmatrix}\begin{pmatrix}e_1\\ e_2\end{pmatrix} \quad (42)$$

Now, If we combine all the above equations, we can have $\dot{V},,\ \tilde{\eta}^T(t)(\Pi\otimes I_n)\tilde{\eta}(t)+\tilde{\eta}^T(t)U^T\mathbf{Q}_2^{-1}U\tilde{\eta}(t)$ where Π is LMIs (8). Due to the considered Lyapunov-Krasovskii candidate, if $\dot{V}(t)<0$ then the origin is asymptotically stable. Therefore, it can be simply moved to LMIs (8).

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